Using Markov Chains to Find Probabilities in Monopoly

Which properties are the best to buy?

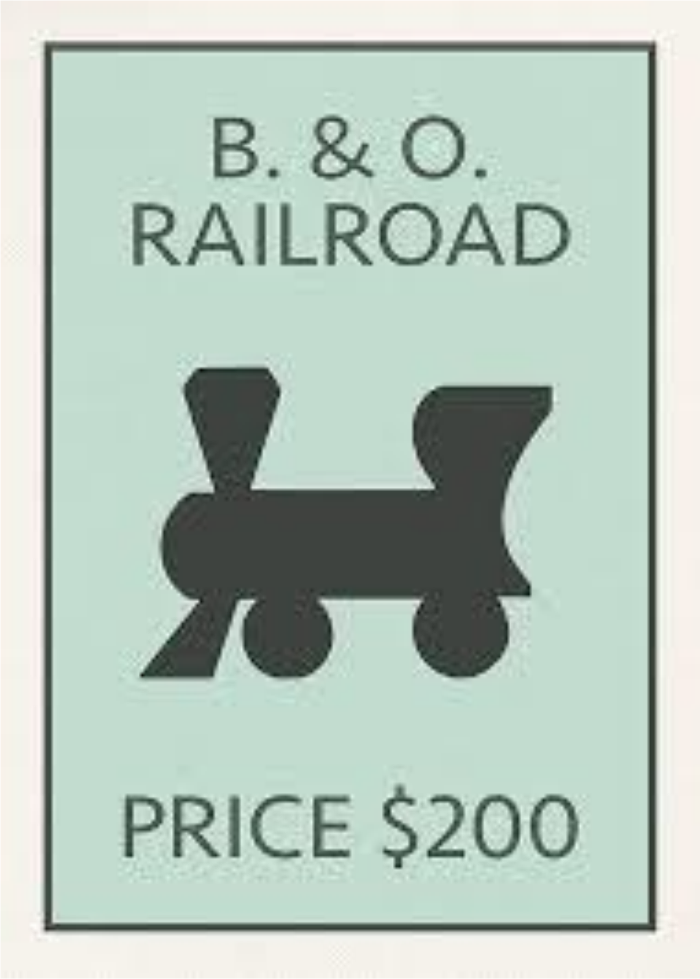
Eleanor Miranda

On Your Turn

- Roll two 6-sided dice.
- Move forward that many spaces.
- Follow the rules for whatever space you land on.

Spaces

- If you land on a property owned by someone else, pay them rent.
- If you land on an unowned property, you may buy it.
- Properties include streets, railroads, and utilities.



- If you land on Luxury/Income Tax, pay the amount shown on the board.
- If you land on a Chance/Community Chest space, choose a card and follow its instructions.

- If you land on Free Parking or Just Visiting, nothing happens.
- If you land on Go To Jail, then you go to jail.

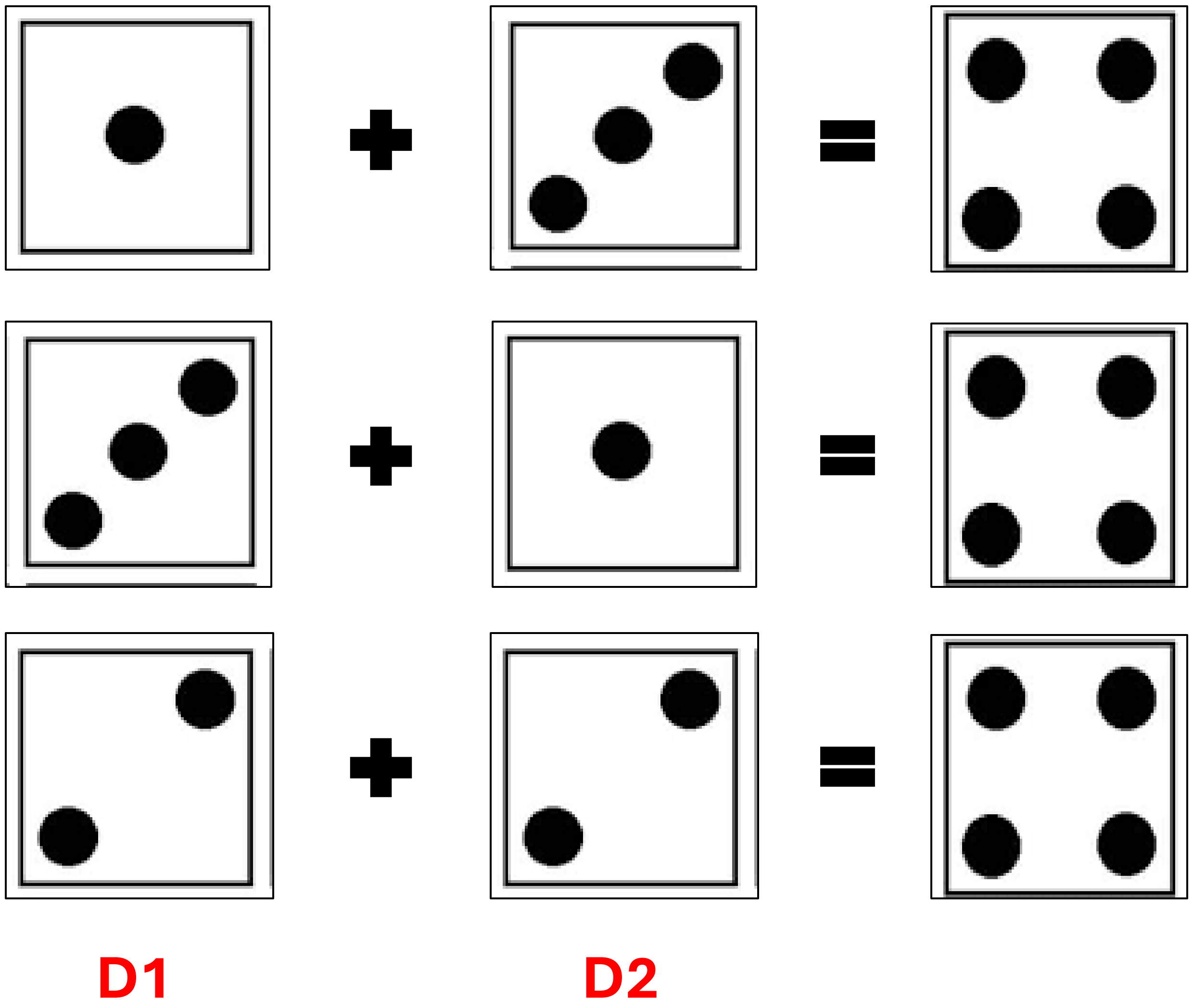
Dice Roll Probabilities

1	2	3	4	5	6	7	8	9	10	11	12
0	$^1/_{36}$	$^2/_{36}$	$^3/_{36}$	$^4/_{36}$	$^5/_{36}$	$^6/_{36}$	$^5/_{36}$	$^4/_{36}$	$^3/_{36}$	$^2/_{36}$	$^1/_{36}$
0%	2.78%	5.56%	8.33%	11.1%	13.9%	16.7%	13.9%	11.1%	8.33%	5.56%	2.78%

Most likely: 7

Least likely: 2 or 12

Example: Rolling a 4



The probability of rolling a 4 is 3 out of 36 possible combinations.

$^{3}/_{36}$

Markov Chain

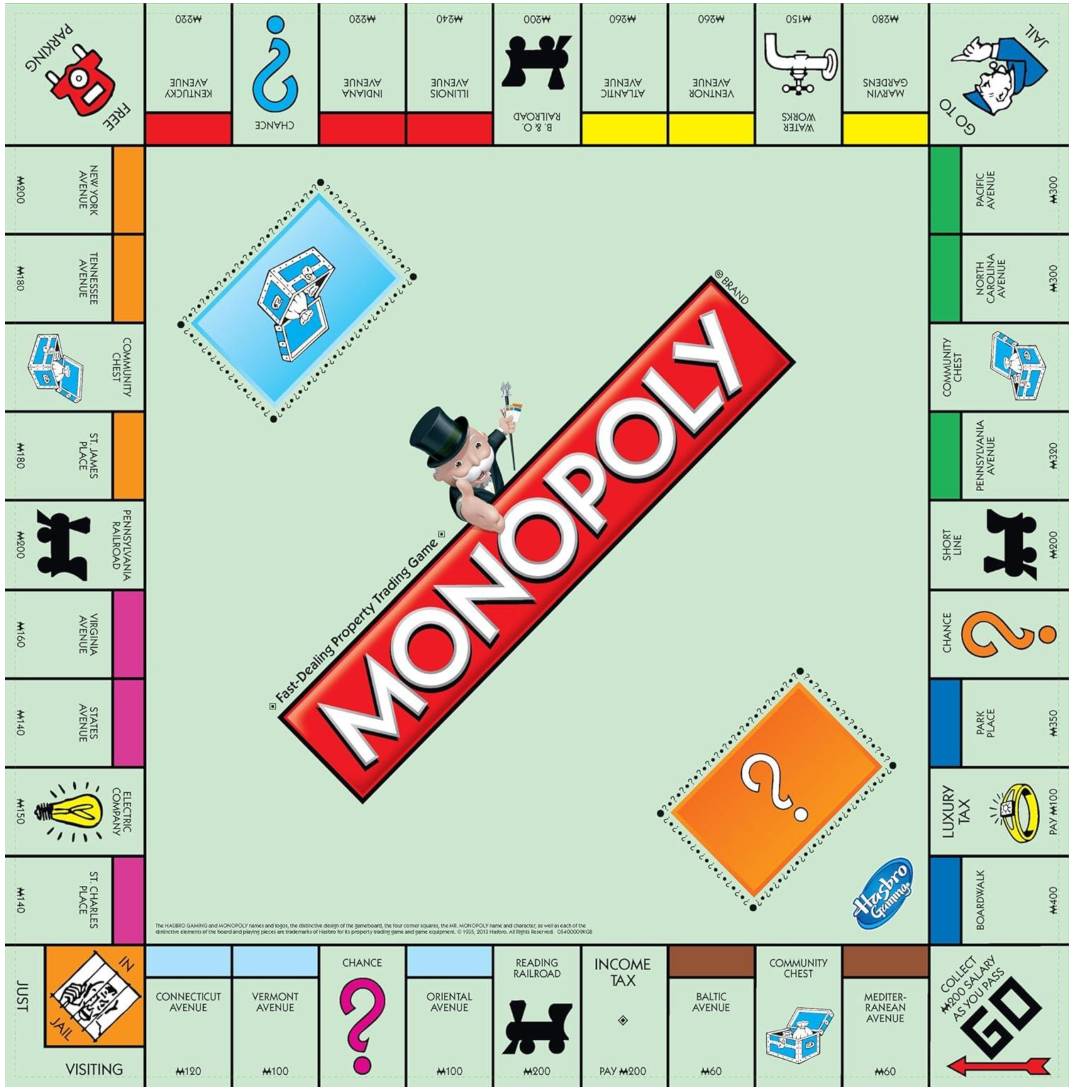
Definition: A sequence of random variables $X_1, X_2, \ldots, X_n$ so that the probability of the next state depends only on the current state and is independent of all previous states.

$$P(H_{n-1} = j | H_n = i, H_{n-1} = i-1, \ldots, H_o = i_o) = P(H_{n-1} = j | H_n = i) = p_{ij}$$

This allows us to create a transition matrix *P* from the transition probability p_{ij} where:

$$p_{ij} = P(H_{n+1} = j | H_n = i), i, j \in S$$
$$\Sigma p_{ij} = 1$$

Transition Matrix



j=4 i=1

- i x j matrix
- i is the starting space
- j is the ending space

- Example: The probability of going from Go → Baltic Ave you would find the value in the 1st row and 4th column of the matrix ($2/36$).
- 40 x 40 matrix (40 spaces)

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} & \frac{1}{36} & 0 \cdots 0 \\ 0 & 0 & 0 & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} & \frac{1}{36} \cdots 0 \\ 0 & 0 & 0 & 0 & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} \cdots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots \end{bmatrix}$$

Rules that Affect the Markov Chain

- Doubles → Roll Again
- 3 Doubles in a row → Jail
- Chance or Community Chest → Follow the card's instructions
  - It could move you to another place on the board
- To get out of jail:
  - Roll a double
    - After 3 turns you must pay $50*
  - Get Out of Jail Free card*
  - Pay $50

*For this analysis we will assume players choose to stay in jail and we will ignore the Get Out of Jail Free cards.

New Transition Matrix

- 120 x 120 matrix
  - Modify the matrix *P* to account for:
    - Position
      - 39 non-jail spaces
    - Consecutive Doubles (0, 1, or 2 rolls)
      - x3 states
    - Time in Jail (0, 1, or 2 turns)
      - +3 stationary turns
    - 39 x 3 + 3 = 120

Short-Term Probabilities

- π_n : short-term distribution which represents the probability of landing on a space after *n* turns

- π_0 is a 1 x 120 vector with 100% probability on the starting space
- Then, we have:

$$\pi_n = \pi_0 P^n$$

Short-Term Example

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \pi_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\text{1 turn: } \pi_1 = \pi_0 \cdot P = \cdots = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

$$\text{2 turns: } \pi_2 = \pi_0 \cdot P^2 = \cdots = \begin{bmatrix} \frac{2}{7} & \frac{1}{6} & \frac{1}{5} \end{bmatrix}$$

$$\text{3 turns: } \pi_3 = \pi_0 \cdot P^3 = \cdots = \begin{bmatrix} \frac{1}{5} & \frac{1}{4} & \frac{2}{9} \end{bmatrix}$$

Long-Term Probabilities: Stationary Distribution

π represents the stationary distribution in which:

$$P = \text{transition matrix} \quad \text{and} \quad \pi = \text{probability vector}$$

$$\pi = \pi P \quad \text{and} \quad \Sigma_i \pi_i = 1 \quad \text{and} \quad \pi_i \geq 0$$

$$\pi = \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \end{bmatrix}$$

Long-Term Example

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1 = 0\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{4}\pi_3 \quad \text{and} \quad \pi_2 = \frac{1}{2}\pi_1 + 0\pi_2 + \frac{2}{3}\pi_3 \quad \text{and} \quad \pi_3 = \frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\pi_3$$

So, $\pi_1 = \frac{8}{37}$, $\pi_2 = \frac{9}{37}$, and $\pi_3 = \frac{20}{37}$.

Thus, $\pi = \begin{bmatrix} \frac{8}{37} & \frac{9}{37} & \frac{20}{37} \end{bmatrix}$.

Expected Return

- The long-term average return from a property per turn, averaged over all possible spaces.

$$E[R] = \sum_{i=1}^{120} \pi_i \cdot r_i$$

π_i: probability of landing on space i

r_i: rent collected when a player lands on a property that is owned

Expected Return Example

$$\pi = \begin{bmatrix} \frac{8}{37} & \frac{9}{37} & \frac{20}{37} \end{bmatrix}$$

Let $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$.

$$E[R] = \pi_1 \cdot r_1 + \pi_2 \cdot r_2 + \pi_3 \cdot r_3$$

$$E[R] = (\frac{8}{37})(1) + (\frac{9}{37})(2) + (\frac{20}{37})(3)$$

$$E[R] = \frac{86}{37} = 2.32$$

Results

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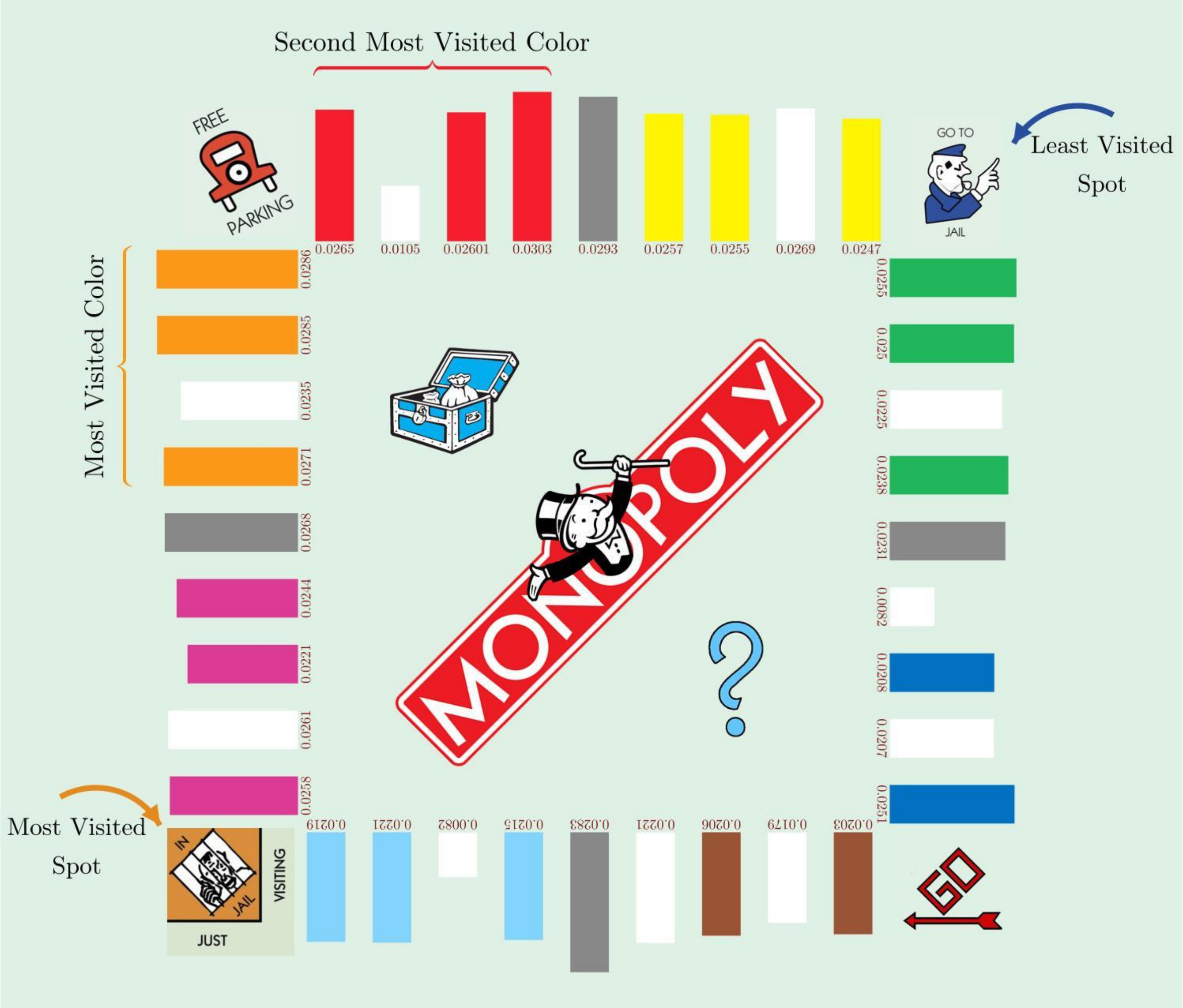
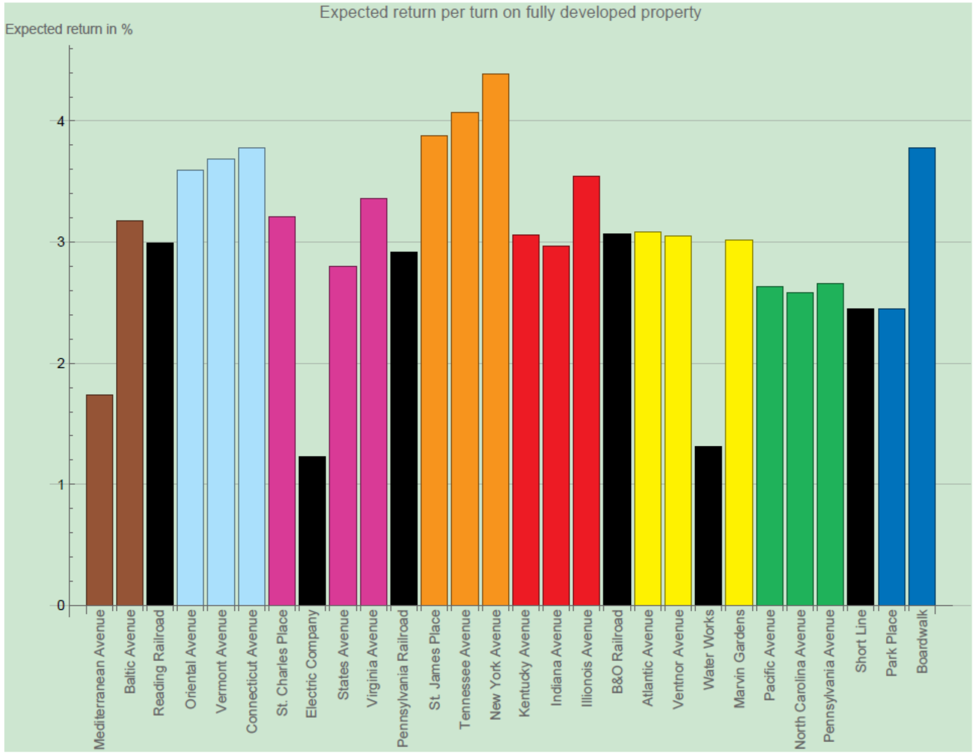
TABLE I. Position Frequencies.

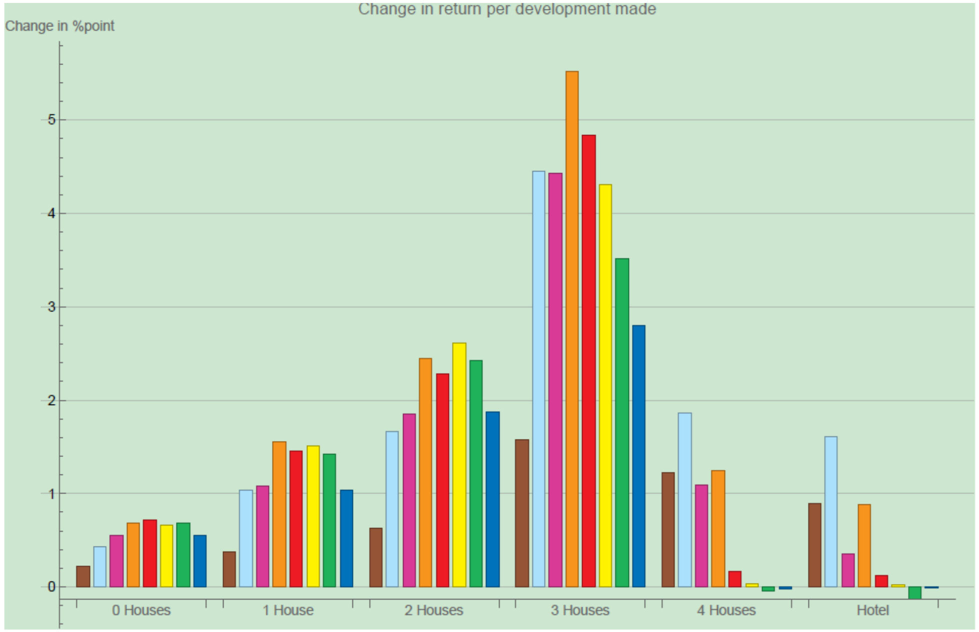
n	Position	Remain in J $p = 0$	Leave J $p = 1$	Normalized state frequencies, $p = .5$ n	n'	n''
0	Go	.0345904	.0368	.0306	.0046	.0007
1	Mediterranean	.0238252	.0252	.0200	.0039	.0007
2	Community Chest	.0210683	.0223	.0187	.0027	.0004
3	Baltic	.0241763	.0256	.0205	.0038	.0007
4	Income Tax	.0260343	.0277	.0226	.0039	.0006
5	Reading RR	.0332449	.0352	.0285	.0050	.0009
6	Oriental	.0253014	.0268	.0222	.0035	.0005
7	Chance	.0096756	.0102	.0083	.0014	.0003
8	Vermont	.0259620	.0274	.0229	.0034	.0005
9	Connecticut	.0257326	.0272	.0226	.0034	.0006
10	Visiting Jail	.0253909	.0269	.0219	.0038	.0006
11	St. Charles	.0303370	.0321	.0263	.0043	.0008
12	Electric Co	.0310260	.0310	.0239	.0064	.0007
13	States	.0258042	.0281	.0231	.0035	.0006
14	Virginia	.0287890	.0293	.0227	.0057	.0007
15	Pennsylvania RR	.0312800	.0346	.0285	.0041	.0007
16	St. James	.0318117	.0331	.0261	.0058	.0007
17	Community Chest	.0272474	.0307	.0255	.0033	.0005
18	Tennessee	.0334833	.0349	.0276	.0059	.0008
19	New York	.0333734	.0366	.0306	.0041	.0006
20	Free Parking	.0335340	.0343	.0270	.0061	.0009
21	Kentucky	.0310290	.0336	.0276	.0044	.0007
22	Chance	.0124010	.0125	.0098	.0023	.0004
23	Indiana	.0304690	.0325	.0265	.0045	.0007
24	Illinois	.0355326	.0377	.0312	.0047	.0010
25	B. and O. RR	.0343378	.0364	.0295	.0053	.0008
26	Atlantic	.0301120	.0321	.0265	.0040	.0008
27	Ventnor	.0298999	.0318	.0256	.0047	.0007
28	Water Works	.0314613	.0333	.0272	.0045	.0009
29	Marvin Gardens	.0289685	.0307	.0247	.0046	.0007
30	Jail total	.1122458	.0469	.0459	.0191	.0080
31	Pacific	.0299534	.0318	.0257	.0046	.0008
32	North Carolina	.0293420	.0312	.0265	.0034	.0006
33	Community Chest	.0264392	.0282	.0226	.0041	.0007
34	Pennsylvania	.0279291	.0297	.0251	.0034	.0005
35	Short Line RR	.0271871	.0289	.0231	.0043	.0008
36	Chance	.0096825	.0102	.0084	.0014	.0002
37	Park Place	.0244447	.0260	.0204	.0042	.0008
38	Luxury Tax	.0243572	.0260	.0217	.0032	.0005
39	Boardwalk	.0294734	.0312	.0248	.0048	.0009

(Bishop & Ash)



Since Jail is the most visited space, **Orange** and **Red** properties have higher expected returns.





- Highest Expected Return for fully developed properties:
  - **Orange**
  - New York Avenue
- Lowest Expected Return for fully developed properties:
  - **Utilities, Brown**
  - Electric Company
- Best time to stop development:
  - 3 houses

(Maatje)

Conclusion

- The Monopoly board is not uniformly distributed.
- There is a unique long-term distribution since the Markov Chain is irreducible.
- The best properties to buy are **Orange** and **Red**.
- It is best to develop properties to 3 houses.
- The worst properties to buy are **Brown** and **Utilities**.

References

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